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Classical distributions of charged dust

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The paper considers the equations of classical hydrodynamics and electromagnetism for a distribution of charged dust. Some general theorems and formulae are obtained.

The problem of cosmology has been discussed previously by many authors on the basis of Newtonian mechanics; and the general conclusion reached that while there exists a close parallelism between Newtonian and relativistic cosmologies, there is one important difference in that Newtonian mechanics allows many models which have no analogues in the relativistic theory (Heckmann & Schucking 1955, Raychaudhuri 1957). In recent years there has grown up a considerable literature on the statics and dynamics of charged dust distributions in general relativity, (De 1968, De & Raychaudhuri 1968, Som & Raychaudhuri 1968, Faulkes 1969, Hamoui 1969, Raychaudhuri & De 1970) and it would be of interest to examine how these results, obtained from general relativity, compare with those from Newtonian theory. Indeed, it was pointed out by Som & Raychaudhuri (1968) that there did not exist any classical analogue of an interesting class of solutions obtained by them.

In this preliminary investigation we present some general formulae by considering the coupled system of classical hydrodynamical equations for pressureless charged dust and Maxwell equations.

We have not introduced the ideas of special relativity even—so that we have not considered the electromagnetic energy density as a source of gravitational field and apparently our results can thus be of significance only for weak enough electromagnetic fields and velocities of dust small compared to that of light.

The following results obtained seem to be of interest :

(i) A formula for the charge density in terms of the electric and magnetic field vectors and the acceleration and vorticity of the dust.

(ii) The result that, in the absence of magnetic field, the vorticity and electric field are orthogonal.

(iii) A theorem that if the magnetic field vanishes—the electric flux through any element of area bounded by particles of the dust is a constant of motion.

(iv) For an irrotational motion in the absence of magnetic fields, the electric field vector is orthogonal to the surfaces defined by constant values of (σ/ρ)

(v) A relation between the characteristics of motions (vorticity, acceleration, expansion and shear) and the matter density.

There exists results closely analogous to the above in the relativistic investigations of Raychaudhuri & De (1970).

The basic equations for a distribution of charged dust are :

$$\dot{\rho}+(\rho v_i),_i = 0 \tag{11}$$

$$\dot{v}_i+v_{i,k}\, v_k = -V,_i+\sigma/\rho(E_i+1/c\, e_{ikl}\, v_k H_l) \tag{2}$$

$$V,_{ii} = 4\pi\rho \tag{3}$$

$$E_{i,i} = 4\pi\sigma \tag{4}$$

$$H_{i,i} = 0 \tag{5}$$

$$e_{ikl}\, E_{l,k} = -\dot{H}_i/c \tag{6}$$

$$e_{ikl}\, H_{l,k} = \dot{E}_i/c+4\pi J_i \tag{7}$$

$$J_i = \sigma v_i \tag{8}$$

$$\tfrac{1}{2}e_{ikl}\, v_{l,k} = \omega_i \tag{9}$$

ρ, σ are the matter and charge densities respectively, $\vec{v}$, the velocity of the dust, $\vec{E}$ and $\vec{H}$ the electric and magnetic field vectors and $\vec{J}$ the current vector. V indicates the Newtonian gravitational potential which satisfies Poisson equation (equation (3)), the comma followed by an index indicates differentiation with respect to that coordinate and the Einstein convention of summing over repeated index is used, e_{ikl} is the Levi-Civita antisymmetric symbol. The conductivity of the charge is neglected and the dielectric constant and permeability are both put equal to unity and $\vec{\omega}$ is the vorticity vector.

From (1) $$\frac{1}{\rho}\,\frac{D\rho}{Dt}+v_{i,i} = 0 \quad \ldots \quad (10)$$

where (D/Dt) signifies differentiation with the fluid

i.e., $$\frac{D}{Dt} = \frac{\partial}{\partial t}+v_i \cdot \frac{\partial}{\partial x_i} \quad \ldots \quad (11)$$

Now let $$v_{i,i} \equiv \frac{3}{G}\,\frac{DG}{Dt} \quad \ldots \quad (12)$$

Then from (10) $$\frac{D}{Dt}\,(\rho G^3) = 0 \quad \ldots \quad (13)$$

Again, on taking divergence of (7) and combining with (4) and (12) we get an equation of conservation of charge *viz.* $D/Dt(\sigma G^3) = 0$ ... (14)

Now the field vectors as observed by an observer moving with the fluid are given by

$$E_i^* = E_i + 1/c.\ e_{ikl} v_k H_l \qquad \dots \quad (15)$$

and

$$H_i^* = H_i - 1/c.\ e_{ikl} v_k E_l \qquad \dots \quad (16)$$

Taking the divergence of (15) and combining with (9) and (7) we get after a little reduction

$$4\pi\sigma = E^*_{i,i} - 2H_i \frac{\omega_i}{c} - \frac{v_i}{c} \frac{\dot{E}_i}{c} \qquad \dots \quad (17)$$

This may be compared with the formula obtained from general relativity

$$4\pi\sigma = E^\alpha{}_{;\alpha} + E^\alpha \dot{v}_\alpha - 2H_\alpha \omega^\alpha \qquad \dots \quad (18)$$

Similarly from(16), (5), (6), (9) we obtain

$$H^*_{i,i} + 2E_i \frac{\omega_i}{c} - \frac{v_i}{c} \frac{\dot{H}_i}{c} = 0 \qquad \dots \quad (19)$$

where again we have the analogous formula in general relativity

$$H^\alpha{}_{;\alpha} + 2E_\alpha w^\alpha + H^\alpha \dot{v}_\alpha = 0 \qquad \dots \quad (20)$$

It may be noted that to our order of approximation $\frac{v_i \dot{H}_i}{c^2}$ may be replaced by $\frac{v_i \dot{H}_i^*}{c^2}$ so that for $H^* = 0$ equation (19) yields

$$E_i \omega_i = 0 \qquad \dots \quad (21)$$

Under the condition of no magnetic field we obtain combining with (7), (4), (12) and after a little reduction

$$G^3 \frac{DE_i}{Dt} = G^3 E_l \frac{\partial v_i}{\partial x_l} - E_i G^3 \frac{3}{G} \frac{DG}{Dt} \qquad \dots \quad (22)$$

or,

$$\frac{D}{Dt}(E_i G^3) = E_l G^3 \frac{\partial v_i}{\partial x_l}$$

or,

$$\frac{D}{Dt}\left(\frac{E_i}{\rho}\right) = \frac{E_l}{\rho} \frac{\partial v_i}{\partial x_l} \qquad \dots \quad (23)$$

Let us now consider two points A and B in the dust lying instantaneously on an electric line of force at a distance apart

$$dx_i = \lambda \frac{E_i}{\rho} \qquad \dots \quad (24)$$

where λ is a small constant. We have for the difference of velocity at A and B for the i-th component

$$v_{iB} - v_{iA} = \frac{\partial v_i}{\partial x_k} dx_k$$

or, with the help of (23) and (24)

$$v_{iB}-v_{iA} = \lambda \frac{D}{Dt}\left(\frac{E_i}{\rho}\right) \tag{25}$$

Now at an instant later by dt, when the particles are at A^1 and B^1 we have

$$dx'_i = dx_i+(v_{iB}-v_{iA})dt$$

or,

$$dx'_i = \lambda\left[\frac{E_i}{\rho}+\frac{D}{Dt}\left(\frac{E_i}{\rho}\right)dt\right] \quad \ldots \tag{26}$$

Equation (26) shows that the relation (24) is not affected by the motion. Hence if S be the area of an element of a tube of force, we have from the conservation of mass $\rho s dx_i =$ constant and from (24), $E_i S =$ constant *i.e.*, the electric intensity varies inversely as the cross-sectional area of an element of fluid orthogonal to the intensity. Considering the curl of equation (2) and combining with (11), (12), (6) and (15) and after a little reduction we obtain

$$\frac{D}{Dt}(v_{\check{ik}}G^2)-\frac{1}{2c}\left(\frac{\sigma}{\rho}\right)\frac{D}{Dt}(H_l G^2)+\frac{G^2}{2}e_{lmn}\left(\frac{\sigma}{\rho}\right)_{,m}E^*_n = 0 \quad \ldots \tag{27}$$

Equation (27) yields in absence of magnetic field and rotation the result that electric field vector is orthogonal to the surfaces defined by constant values of (σ/ρ).

For the relation (v) we take the divergence of (2) and combine with (3), (4), (7), (9), (12) and after a little reduction we obtain

$$\frac{3}{G}\frac{D^2G}{Dt^2} = -4\pi\rho\left(1-\frac{\sigma^2}{\rho^2}\right)+\left(\frac{\sigma}{\rho}\right)_{,i}E_i+\left(\frac{\sigma}{\rho}\right)_{,i}e_{ikl}v_k H_l+2\left(\frac{\sigma}{\rho}\right)H_i\frac{\omega_i}{c}$$

$$-\left(\frac{\sigma}{\rho}\right)\frac{v_i}{c}\frac{\dot{E}_i}{c}-v_{i,k}v_{k,i}+\frac{1}{3}(v_{i,i})^2 \quad \ldots \tag{28}$$

Writing,

$$v_{i,k} = v_{\underline{ik}}+v_{\check{ik}}$$

$$v_{k,i} = v_{\underline{ik}}-v_{\check{ik}}$$

we have

$$\frac{1}{3}(v_{i,i})^2-v_{i,k}v_{k,i} = -\phi^2+2\omega^2 \quad \ldots \tag{29}$$

where again shear (ϕ^2) vanishes if and only if $v_{ik} = \alpha\delta_{ik}$, *i.e.*, the expansion be isotropic at the point considered and is positive otherwise. We have from (28), (29) and (15) finally,

$$\frac{3}{G}\frac{D^2G}{Dt^2} = -4\pi\rho\left(1-\frac{\sigma^2}{\rho^2}\right)+\left(\frac{\sigma}{\rho}\right)_{,i}E^*_i+2\left(\frac{\sigma}{\rho}\right)H_i\frac{\omega_i}{c}-\phi^2+2\omega^2-\left(\frac{\sigma}{\rho}\right)\frac{v_i}{c}\frac{\dot{E}_i}{c} \quad \ldots \tag{30}$$

This may be compared with the general relativity result

$$4\pi\rho\left(1-\frac{\sigma^2}{\rho^2}\right)-E^2\left(1-\frac{\sigma^2}{\rho^2}\right)-H^2 = -2\left(\frac{\sigma}{\rho}\right)H^\alpha\omega_\alpha+2(\phi^2-\omega^2)$$

$$+\frac{\theta^2}{3}+\theta_{,\alpha}v^\alpha+\left(\frac{\sigma}{\rho}\right)_{,\alpha}E^\alpha \tag{31}$$

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